ON CERTAIN BOUNDARY-VALUE PROBLEMS FOR THE EQUATIONS OF SEEPAGE OF A LIQUID IN FISSURED ROCKS

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It has been shown in [1,2] that the equations of seepage of a uniform, slightly compressible liquid through fissured rocks is of the form

$$\eta \Delta p_1 + p_2 - p_1 = 0, \qquad \eta \frac{\partial p_2}{\partial t} + \varkappa (p_2 - p_1) = 0 \tag{1}$$

so that the pressure of the liquid in the fissures p_1 and in the pores p_2 satisfy similar equations

$$Lp_1 = 0, \qquad Lp_2 = 0, \qquad L \equiv \frac{\partial}{\partial t} - \eta \frac{\partial \Delta}{\partial t} - \varkappa \Delta \tag{2}$$

where Δ represents the Laplace operator, η and κ are characteristics of the rock and of the liquid, as given in [1,2]. In [2] it was shown that if the initial distribution of the quantity p which satisfies the equation

$$Lp = 0 \tag{3}$$

has discontinuities of the first kind in the function itself, or in its derivative with respect to some direction n, then these discontinuities do not disappear instantaneously, as in the case of the equation of heat conduction, but decrease with time according to the law

$$[p] = [p]_{t=0} e^{-\varkappa t/\eta}, \qquad \left[\frac{\partial p}{\partial n}\right] = \left[\frac{\partial p}{\partial n}\right]_{t=0} e^{-\varkappa t/\eta} \tag{4}$$

(where, as usual, the square brackets denote the step in the enclosed quantity). At approximately the same time similar results were obtained by Nemchinov [3] in connection with a different physical problem. There is an error in [2] which, the author understands, was discovered by P.P. Zolotarev; namely that in the solution of the boundary-value problems studied in this paper for equation (3), reference was made, in error, to the pressure p_1 in the fissures, whereas the solution, as well as the investigation of the discontinuities, should refer to the pressure p_2 in the pores.

One of the boundary-value problems requires that the flow of liquid is specified on the boundary of the volume of rock. Since the permeability of the blocks is ignored, this flow is equal to $(k_1/\mu)\partial p_1/\partial n$ (formula (1.1) in [2]). From this and from the second of equations (1) we obtain the following condition for the case when the flow on the boundary of the volume of rock is given:

$$-\left\{\frac{\eta}{\varkappa}\frac{\partial}{\partial t}\left(\frac{k_1}{\mu}\frac{\partial p_2}{\partial n}\right)+\frac{k_1}{\mu}\frac{\partial p_2}{\partial n}\right\}=f(S,t) \tag{5}$$

which differs from the corresponding condition (4.4) in [2] only in that it contains p_2 instead of p_1. One other boundary-value problem also requires the pressure p_2 to be specified on the outside of the boundary of the volume of rock; the boundary condition for the third problem is a linear combination of the boundary conditions for the first two.

In order to clarify this question let us consider in more detail the presentation of the basic problems for the system of equations of seepage in fissured rocks (1). The system (1) is the limiting case of a system of equations of motion of a liquid in a medium with dual porosity [1,2]

$$\eta\Delta p_1=\varepsilon_1\frac{\partial p_1}{\partial t}-(p_2-p_1),\qquad \varepsilon_2\Delta p_2=\eta\frac{\partial p_2}{\partial t}+\varkappa(p_2-p_1) \tag{6}$$

corresponding to ε_1 and ε_2 equal to zero. For our present purposes it is sufficient to consider the case of a negligibly small permeability of the blocks, setting $\varepsilon_2 = 0$ in (6); this system can then be written in the form

$$\eta\Delta p_1=\varepsilon_1\frac{\partial p_1}{\partial t}-(p_2-p_1),\qquad \eta\frac{\partial p_2}{\partial t}+\varkappa(p_2-p_1)=0 \tag{7}$$

The Cauchy problem and the boundary-value problems for the system (1) must be the limiting case of the corresponding problems for the system (7) as $\varepsilon_1 \to 0$.

For $\varepsilon_1 \neq 0$ the system (7) is correct in the sense of Petrovskii [4] with a correctness index zero. This means, in particular, that the initial distributions of p_1 and p_2 in the Cauchy problem can be specified independently, they can have discontinuities of the first kind and must be subject to certain restrictions on increase to infinity. As $\varepsilon_1 \to 0$ the solution of system (7) tends to the corresponding solution of system (1) over any finite interval of time $0 < t_1 < t < T < \infty$. Within the interval $0 < t < t_1$, where t_1 is a fixed number, no matter how small, the behavior of the solution as $\varepsilon_1 \to 0$ can be investigated by the boundary-layer method for linear equations developed by Vishik and Liusternik [6]. According to this method a "fast" time $\theta = t/\varepsilon_1$ is introduced into the system (7) and this system can then be re-written as

$$\eta \Delta p_1 = \frac{\partial p_1}{\partial \theta} - (p_2 - p_1), \qquad \eta \frac{\partial p_2}{\partial \theta} + \varkappa \varepsilon_1 (p_2 - p_1) = 0 \tag{8}$$

If we now let ε_1 tend to zero, we find that $\theta_1 = t_1/\varepsilon_1 \to \infty$, $\partial p_2/\partial\theta \to 0$, and consequently when $\theta = \theta_1$, $\partial p_1/\partial\theta \to 0$ also, so that at $t = t_1$ system (8) reduces to the form

$$\eta \Delta p_1 + p_2 - p_1 = 0, \qquad p_2 = p_2(x, y, z, 0) \tag{9}$$

Bearing in mind that we can make t_1 as small as we like, we observe that for the initial distribution of p_1 for system (1) we must take the function defined by the first of equations (9). Thus, in contrast to the Cauchy problem for system (7), the initial distributions of p_1 and p_2 in the Cauchy problem for system (1) cannot be specified independently. It follows from the first of equations (9) that for the functions p_1 and p_2 to be regular [6], i.e. in this case for them not to have singularities of the delta function type and its derivative, the initial distributions of p_1 must be a once continuously differentiable function, whereas for p_2 they can be discontinuous. Let us turn now to the boundary-value problems. For the existing methods of measurement the following types of boundary conditions are of interest: on the outer face of the boundary of the volume of rock under consideration we are given: (1) the pressure of the liquid, (2) the flow, (3) a linear combination of the pressure and the flow.

In order to solve the boundary-value problem we must transfer from the data on the outside of the boundary of the volume of rock under consideration to the corresponding data on its inside. Once again we treat system (1) as the limiting case of system (7). For system (7) with $t = 0$ the pressures p_1 and p_2 on the inside surface of the boundary of the volume of rock are not in general the same. In the previously considered short interval of time $0 < t < t_1$, for the duration of which we can use the system (1), it has been shown that in the limit as $\varepsilon_1 \to 0$ the pressure p_2 does not vary, but remains equal to its initial value at all points in the rock. The pressure p_1, however, satisfies a parabolic equation (the first of equations (8)). Therefore, if at $t = 0$ the liquid pressure p_1 in the fissures at the boundary is not equal to the pressure specified at at boundary, then this jump instantly vanishes, so that by the time $t = t_1$ the distribution of p_1 on the boundary is continuous. For the same reasons by the time $t = t_1$ it is continuous also within the volume of rock. Since by virtue of the negligibly small permeability of the blocks the flow is defined fully by the distribution of p_1, the same applies for the flow and for a linear combination of the flow with the pressure. As $\varepsilon_1 \to 0$ the magnitude of t_1 can be made as small as we like, and therefore these conclusions refer to the initial instant for system (1).

Thus, for system (1) the formulation of the boundary-value problems of the kinds discussed above consists of the following. We specify the initial distribution of p_2, which can be discontinuous and can increase in some way to infinity. From this we find the initial distribution of p_1 as the integral of the first of equations (9) with the boundary conditions of problems (1), or (2), or (3) corresponding to the initial instant. With these initial distributions and boundary conditions system (1) is solved for successive instants of time. In an actual solution of the problem it is more convenient to use equations (3), eliminating one of the unknowns and expressing the boundary and initial conditions in terms of an unknown quantity. The other unknown can be easily found from the second of equations (1). If the initial distribution of p_2 or its derivative with respect to some direction $\partial p_2/\partial n$ has discontinuities, then they vary with time according to formulas (4). The distribution of p_1 is continuous and is once continuously differentiable.

The formulation of the basic boundary-value problems (1) and (2) in terms of p_1 and p_2 is as follows:

$$p_1(S^+) = p_1(S^-) = p_2(S^+) = f(S, t) \tag{10}$$

$$\frac{\partial p_1(S^+)}{\partial n} = \frac{\partial p_1(S^-)}{\partial n} = \frac{1}{\varkappa}\left\{\eta \frac{\partial}{\partial t}\left[\frac{\partial p_2(S^-)}{\partial n}\right] + \varkappa \frac{\partial p_2(S^-)}{\partial n}\right\} = g(S, t) \tag{11}$$

(where S^+ and S^- represent the external and internal faces of the boundary of the volume of rock; f and g are given functions).

Ban [7, 8] investigated the problem of determining the parameters of the stratum by extending to fissured rocks a method proposed in [9] for porous rocks based on the application of Laplace transforms. Ban's method is based on the explicit expression of the quantity ψ – the ratio of the Laplace transform of the pressure p on the wall of a well to the Laplace transform of the quantity $(r\partial p/\partial r)$ which determines the inflow into the well. It can easily be seen that ψ does not depend on which pressure forms the basis of the calculations, the pore pressure or the pressure in the fissures. Indeed, from the second of equations (1) we have

$$p_1 = \frac{\eta}{\varkappa}\frac{\partial p_2}{\partial t} + p_2, \qquad \left(r\frac{\partial p_1}{\partial r}\right) = \frac{\eta}{\varkappa}\frac{\partial}{\partial t}\left(r\frac{\partial p_2}{\partial r}\right) + \left(r\frac{\partial p_2}{\partial r}\right) \tag{12}$$

Taking the Laplace transforms of both of these formulas, with the parameter $s = 1/\vartheta$, we obtain

$$U_1 = \left(\frac{\eta}{\varkappa\vartheta} + 1\right)U_2, \qquad \left(r\frac{\partial U_1}{\partial r}\right) = \left(\frac{\eta}{\varkappa\vartheta} + 1\right)\left(r\frac{\partial U_2}{\partial r}\right)$$

where U_1 and U_2 are the Laplace transforms of p_1 and p_2 respectively, from which we find that

$$\psi = U_1\left(r\frac{\partial U_1}{\partial r}\right)^{-1} = U_2\left(r\frac{\partial U_2}{\partial r}\right)^{-1}$$

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